

ALGEBRAIC TREATMENT OF COLLECTIVE EXCITATIONS IN BARYON SPECTROSCOPY

R. Bijker¹ and A. Leviatan²

¹ R.J. Van de Graaff Laboratory, University of Utrecht,
P.O. Box 80000, 3508 TA Utrecht, The Netherlands

² Racah Institute of Physics, The Hebrew University,
Jerusalem 91904, Israel

INTRODUCTION

Algebraic methods have been used extensively in hadronic physics for the description of the internal degrees of freedom (flavor-spin-color) [1]. Spectrum generating algebras and dynamic symmetries have been very instrumental in the classification of hadronic states and the construction of mass formulas, such as the Gell-Mann-Okubo mass formula [2].

Recently Iachello suggested [3] to use algebraic methods for the geometric structure of hadrons as well, and thus, by combining the degrees of freedom of the internal and relative motion, to obtain a fully algebraic description of hadrons. The key ingredient in such an approach is the choice of a spectrum generating algebra (SGA) for the relevant spatial degrees of freedom, *e.g.* relative quark coordinates or geometric shape variables. The operators representing physical observables, such as masses, are then expressed in terms of elements of the algebra. All matrix elements of interest can be calculated exactly without having to make further approximations. As a first application of this approach the string-like configuration of a quark (q) and an antiquark (\bar{q}) in a meson was described in terms of a $U(4)$ SGA [4, 5]. The $U(4)$ algebra is realized in terms of bosons whose mutual interactions simulate the dynamics of the radius vector connecting the quark and antiquark. This model corresponds to a collective string-like description of mesons.

The situation for baryons is more complex. In a valence quark model, baryons are made up of three quarks (qqq), some of which may be identical. Therefore, unlike the situation in mesons, for baryons one has to construct states with good permutation symmetry. To fulfill the Pauli principle only particular combinations of space, spin-flavor and color representations are allowed. In quark potential models, either nonrelativistic

[6] or in a relativized form [7], baryons are described as a system of quarks interacting through two-body interactions. Most calculations are done in a harmonic oscillator basis and usually only a few oscillator shells are taken into account exactly. As an alternative to such a single-particle type of description, it is of interest to consider the possibility of a collective description of baryons. The quarks participating in a collective motion are strongly correlated and move coherently. In terms of a harmonic oscillator basis collectivity requires the mixing of many different oscillator shells. This is in analogy to the case encountered in nuclei for which a description of α -clustering requires mixing of many shells in the nuclear shell model. In any collective description of baryons one has to mix different oscillator shells in such a way that the permutation symmetry of the quarks is preserved. Furthermore, in order to be able to compare the single-particle with the geometric-collective description of baryons, it would be beneficial to have a framework that encompasses both points of view.

In this contribution we introduce a $U(7)$ spectrum generating algebra as a new algebraic string-like model of baryons. This model satisfies all the above mentioned requirements for a collective description. We show explicitly that in the $U(7)$ model it is possible to incorporate the permutation symmetry of the three quarks exactly and to accommodate both single-particle and collective types of motion. The model space consists of many oscillator shells, but since it is finite dimensional, the $U(7)$ SGA provides a tractable computational scheme, that involves only the diagonalization of finite dimensional matrices. A great advantage is that within the assumptions of the model all calculations can be done exactly. We discuss some special cases in which we derive closed analytic solutions that give a clear insight into the properties of the model. Finally we note that the same $U(7)$ SGA can be used to describe rotations and vibrations in triatomic molecules [8].

THE $U(7)$ QUARK MODEL

The hamiltonian in quark potential models contains a kinetic energy term, a confining potential and the hyperfine interaction. The confining potential is written as the sum of two uncoupled harmonic oscillators in the Jacobi coordinates and a residual two-body interaction. The two relative Jacobi coordinates are

$$\begin{aligned}\vec{\rho} &= (\vec{r}_1 - \vec{r}_2)/\sqrt{2}, \\ \vec{\lambda} &= (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)/\sqrt{6},\end{aligned}\tag{1}$$

where \vec{r}_i is the coordinate of the i -th quark. The group structure associated with the harmonic oscillator part is given by the chain,

$$U(6) \supset U_\rho(3) \otimes U_\lambda(3) \supset SO_\rho(3) \otimes SO_\lambda(3) \supset SO_{\rho\lambda}(3) .\tag{2}$$

The unperturbed eigenvalues are given in terms of $n = n_\rho + n_\lambda$, the total number of quanta in the ρ and λ oscillators. The group chain in eq. (2) provides a convenient basis to diagonalize the full hamiltonian. Typical calculations involve only a few oscillator shells. For a given harmonic oscillator shell, n , there exist well known techniques to construct states of good permutation symmetry, angular momentum and parity [9].

Collective behavior of quarks inside baryons corresponds to a coherent motion of the quarks. A description of such correlated motion in a harmonic oscillator basis requires strong coupling of many different oscillator shells. In order to accommodate such mixing we discuss an extension of the $U(6)$ symmetry group of the harmonic oscillator. From experience gained with algebraic models in nuclear [10] and molecular physics [11], it is

known that this can be achieved by embedding the $U(6)$ algebra in the compact SGA of $U(7)$. This is done by adding a scalar boson under the restriction that the total number of bosons is conserved. The last condition guarantees that the $U(7)$ SGA still describes only six degrees of freedom, which can be associated with the two relative Jacobi coordinates.

The building blocks of the $U(7)$ model are the six components of two dipole bosons with $L^\pi = 1^-$, denoted in second quantization by p_ρ^\dagger and p_λ^\dagger , and a scalar boson with $L^\pi = 0^+$, denoted by s^\dagger . A mass operator that conserves the total number of quanta $N = n_s + n_\rho + n_\lambda$ can be expressed in terms of the 49 generators of $U(7)$,

$$\begin{aligned} D_\lambda &= (p_\lambda^\dagger s - s^\dagger \tilde{p}_\lambda)^{(1)} , & A_\lambda &= i (p_\lambda^\dagger s + s^\dagger \tilde{p}_\lambda)^{(1)} , & G_\lambda^{(l)} &= (p_\rho^\dagger \tilde{p}_\rho - p_\lambda^\dagger \tilde{p}_\lambda)^{(l)} , \\ D_\rho &= (p_\rho^\dagger s - s^\dagger \tilde{p}_\rho)^{(1)} , & A_\rho &= i (p_\rho^\dagger s + s^\dagger \tilde{p}_\rho)^{(1)} , & G_\rho^{(l)} &= (p_\rho^\dagger \tilde{p}_\lambda + p_\lambda^\dagger \tilde{p}_\rho)^{(l)} , \\ G_S^{(l)} &= (p_\rho^\dagger \tilde{p}_\rho + p_\lambda^\dagger \tilde{p}_\lambda)^{(l)} , & G_A^{(l)} &= i (p_\rho^\dagger \tilde{p}_\lambda - p_\lambda^\dagger \tilde{p}_\rho)^{(l)} , & \hat{n}_s &= s^\dagger s , \end{aligned} \quad (3)$$

with $l = 0, 1, 2$ and $\tilde{p}_m = (-1)^{1-m} p_{-m}$. All many-boson states are classified according to the totally symmetric representation $[N]$ of $U(7)$. The value of N determines the number of states in the model space. In view of confinement we expect N to be large. The model space contains several oscillator shells. For a given value of N , it consists of oscillator shells with $n = n_\rho + n_\lambda = 0, 1, \dots, N$. In order to construct a mass operator that properly takes into account the permutation symmetry among the three quarks inside a baryon, we study the transformation properties of the generators of eq. (3) under the permutation group.

PERMUTATION SYMMETRY IN THE $U(7)$ MODEL

The harmonic oscillator basis is very well suited to construct a set of basis states that in addition to good angular momentum and parity also have good permutation symmetry under the interchange of any of the three quarks. For baryons with strangeness -1 or -2 the only permutation that is involved is the interchange of the two identical quarks, $P(12)$, which without loss of generality are taken to be the first two. For baryons with strangeness 0 or -3 the three quarks are indistinguishable and therefore we have to use in addition to $P(12)$ also the cyclic permutation $P(123)$. All other permutations can be expressed in terms of these two elementary ones. Although the $U(7)$ model can describe both strange and nonstrange baryons, in the present contribution we discuss only the nonstrange sector, namely the nucleon and the delta resonances. The spatial wave functions then have to be combined with a spin-flavor and a color part so that the full wave function of the baryon is antisymmetric.

In the current approach the spatial part of the wave function is described in terms of the $U(7)$ SGA. The eigenstates are required to have well defined transformation properties under the permutation group, or equivalently, the mass operator for nonstrange baryon resonances has to be invariant under S_3 . The transformation properties under S_3 of all operators in the model follow from those of s^\dagger , p_ρ^\dagger and p_λ^\dagger ,

$$\begin{aligned} P(12) \begin{pmatrix} s^\dagger \\ p_{\rho,m}^\dagger \\ p_{\lambda,m}^\dagger \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s^\dagger \\ p_{\rho,m}^\dagger \\ p_{\lambda,m}^\dagger \end{pmatrix} , \\ P(123) \begin{pmatrix} s^\dagger \\ p_{\rho,m}^\dagger \\ p_{\lambda,m}^\dagger \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\pi/3 & \sin 2\pi/3 \\ 0 & -\sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix} \begin{pmatrix} s^\dagger \\ p_{\rho,m}^\dagger \\ p_{\lambda,m}^\dagger \end{pmatrix} . \end{aligned} \quad (4)$$

The s -boson is a scalar under the permutation group. There are three different symmetry classes for the permutation of three objects: a symmetric one, S , an antisymmetric

Operator	$P(12)$	$P(123)$	S_3	Young tableau
s^\dagger , $s^\dagger s^\dagger$, $(p_\rho^\dagger p_\rho^\dagger + p_\lambda^\dagger p_\lambda^\dagger)^{(l')}$, \hat{n}_s , $G_S^{(l)}$	1	1	S	$\boxed{1123}$
p_λ^\dagger , $s^\dagger p_\lambda^\dagger$, $(p_\rho^\dagger p_\rho^\dagger - p_\lambda^\dagger p_\lambda^\dagger)^{(l')}$, D_λ , A_λ , $G_\lambda^{(l)}$	1	λ	M_λ	$\boxed{\begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix}}$
p_ρ^\dagger , $s^\dagger p_\rho^\dagger$, $(p_\rho^\dagger p_\lambda^\dagger + p_\lambda^\dagger p_\rho^\dagger)^{(l')}$, D_ρ , A_ρ , $G_\rho^{(l)}$	-1	ρ	M_ρ	$\boxed{\begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix}}$
$(p_\rho^\dagger p_\lambda^\dagger)^{(1)}$, $G_A^{(l)}$	-1	1	A	$\boxed{\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}}$

Table 1: Transformation properties of linear and bilinear operators under the S_3 permutation group. Here $l = 0, 1, 2$ and $l' = 0, 2$.

one, A , and a two-dimensional one of mixed symmetry type, denoted by M_ρ and M_λ . The latter have the same transformation properties as the creation operators, p_ρ^\dagger and p_λ^\dagger . Alternatively, the three symmetry classes can be labeled by the irreducible representations of the point group D_3 (which is isomorphic to S_3) as A_1 , A_2 and E , respectively. It is now straightforward to find the bilinear combinations of creation and annihilation operators that transform irreducibly under the permutation group. The results are shown in table 1.

Next one can use the multiplication rules for S_3 to construct all rotationally invariant interactions that preserve parity and that are scalars under S_3 . As a result we find two terms linear in the $U(7)$ generators of eq. (3),

$$\hat{n}_s \text{ , } \hat{n} = \sqrt{3} G_S^{(0)} \text{ ,} \quad (5)$$

where $\hat{n}_s = \hat{N} - \hat{n}$ is the number operator for scalar bosons, and $\hat{n} = \hat{n}_\rho + \hat{n}_\lambda$ counts the total number of ρ and λ bosons. In addition we have the following quadratic multipole operators,

$$\begin{aligned} &\hat{n}_s \hat{n}_s \text{ , } \hat{n}_s \hat{n} \text{ , } D_\rho \cdot D_\rho + D_\lambda \cdot D_\lambda \text{ , } A_\rho \cdot A_\rho + A_\lambda \cdot A_\lambda \text{ ,} \\ &G_S^{(l)} \cdot G_S^{(l)} \text{ , } G_A^{(l)} \cdot G_A^{(l)} \text{ , } G_\rho^{(l)} \cdot G_\rho^{(l)} + G_\lambda^{(l)} \cdot G_\lambda^{(l)} \text{ .} \end{aligned} \quad (6)$$

We note that not all terms in eqs. (5,6) are independent. A set of independent two-body interactions that are scalars under the S_3 permutation group can be obtained by transforming to normal ordered form,

$$\begin{aligned} &s^\dagger s^\dagger s s \text{ , } s^\dagger (p_\rho^\dagger \cdot \tilde{p}_\rho + p_\lambda^\dagger \cdot \tilde{p}_\lambda) s \text{ , } (p_\rho^\dagger p_\lambda^\dagger)^{(1)} \cdot (\tilde{p}_\lambda \tilde{p}_\rho)^{(1)} \text{ ,} \\ &(p_\rho^\dagger \cdot p_\rho^\dagger + p_\lambda^\dagger \cdot p_\lambda^\dagger) s s + s^\dagger s^\dagger (\tilde{p}_\rho \cdot \tilde{p}_\rho + \tilde{p}_\lambda \cdot \tilde{p}_\lambda) \text{ ,} \end{aligned}$$

$$\begin{aligned}
& (p_\rho^\dagger p_\rho^\dagger - p_\lambda^\dagger p_\lambda^\dagger)^{(l')} \cdot (\tilde{p}_\rho \tilde{p}_\rho - \tilde{p}_\lambda \tilde{p}_\lambda)^{(l')} + 4 (p_\rho^\dagger p_\lambda^\dagger)^{(l')} \cdot (\tilde{p}_\lambda \tilde{p}_\rho)^{(l')} , \\
& (p_\rho^\dagger p_\rho^\dagger + p_\lambda^\dagger p_\lambda^\dagger)^{(l')} \cdot (\tilde{p}_\rho \tilde{p}_\rho + \tilde{p}_\lambda \tilde{p}_\lambda)^{(l')} ,
\end{aligned} \tag{7}$$

with $l' = 0, 2$.

Diagonalization of a general S_3 -invariant mass operator composed of the terms in eqs. (5-7) yields wave functions of good permutation symmetry. The transformation properties of these wave functions under S_3 can be determined by taking the overlap with harmonic oscillator wave functions that by construction have good (and known) permutation symmetry [9]. We used a more direct method which is also based on the transformation properties under $P(12)$ and $P(123)$. The symmetry classes (S, M_λ) are distinguished from (A, M_ρ) by a choice of basis states with the number of quanta in the ρ oscillator, n_ρ , even or odd, respectively. The classes (S, A) are distinguished from (M_λ, M_ρ) by the expectation value K_y^2 (with K_y integer) of the operator, $\hat{K}_y^2 = 3 G_A^{(0)} G_A^{(0)}$, namely, $|K_y| = 0, 3, 6, \dots$ for the former and $|K_y| = 1, 2, 4, 5, \dots$ for the latter. Formally K_y corresponds to the quantum number, m , that was used in ref. [12] to classify the states in the quark potential model. It is important to note that our method is valid for any oscillator shell.

GEOMETRIC SHAPE AND ITS EXCITATIONS

The previous analysis shows that already at the level of one- and two-body interactions, there are many possible terms that can be included in a S_3 -invariant mass operator in the $U(7)$ model. In this section we analyze the model in terms of geometric variables and study its elementary excitations to gain a better understanding of the physical content of each of the allowed interaction terms. This will provide a selection criterion which terms to include in the mass operator.

A geometric interpretation of the algebraic model can be obtained by means of a coherent state [13], which for the $U(7)$ model takes the form of a condensate of N bosons,

$$|N; c\rangle = \frac{1}{\sqrt{N!}} (b_c^\dagger)^N |0\rangle, \tag{8}$$

with

$$b_c^\dagger = (1 + R^2)^{-1/2} \left[s^\dagger + r_\rho p_{\rho,0}^\dagger + r_\lambda \sum_m d_{m,0}^{(1)}(\theta) p_{\lambda,m}^\dagger \right]. \tag{9}$$

Here $R = \sqrt{r_\rho^2 + r_\lambda^2}$. The two vectors, \vec{r}_ρ and \vec{r}_λ , in the condensate span the xz -plane. We have chosen the z -axis along the direction of $\vec{r}_\rho = r_\rho \hat{z}$, and \vec{r}_λ is rotated by an angle θ about the out-of-plane y -axis, $\vec{r}_\rho \cdot \vec{r}_\lambda = r_\rho r_\lambda \cos \theta$.

The expectation value of the mass operator in the condensate defines a classical potential function, $V(r_\rho r_\lambda \theta) = \langle N; c | \hat{M}^2 | N; c \rangle$. The equilibrium shape is determined by minimizing the potential function with respect to the coordinates, r_ρ and r_λ , and the relative angle θ . The equilibrium values of these shape parameters are denoted by \bar{r}_ρ , \bar{r}_λ and $\bar{\theta}$, respectively. For the problem at hand, three identical quarks inside a baryon, the most general S_3 -invariant $U(7)$ mass operator with one- and two-body terms yields a rigid nonlinear equilibrium shape characterized by

$$\bar{r}_\rho = \bar{r}_\lambda, \quad \bar{\theta} = \frac{\pi}{2}. \tag{10}$$

These are precisely the conditions satisfied by the Jacobi coordinates of eq. (1) for an equilateral triangular shape. This strongly suggests to associate these coordinates

with the algebraic shape parameters in eq. (9). The nonnegative equilibrium value $\bar{r}_\rho = \bar{r}_\lambda = \bar{r}$ is a measure of the dipole deformation in the ground state condensate. For the special case of $\bar{r} = 0$ we recover the spherical condensate of the quark potential model.

The equilibrium configuration of three identical quarks in a baryon is represented in the $U(7)$ model by an intrinsic state, eq. (8), composed of N condensate bosons with $r_\rho = r_\lambda$ and $\theta = \pi/2$:

$$b_c^\dagger = (1 + R^2)^{-1/2} \left\{ s^\dagger + R \frac{1}{\sqrt{2}} \left[p_{\rho,0}^\dagger - \frac{1}{\sqrt{2}} (p_{\lambda,1}^\dagger - p_{\lambda,-1}^\dagger) \right] \right\}. \quad (11)$$

It corresponds geometrically to an equilateral triangular shape with the y -axis as a threefold symmetry axis. Excitations of the equilibrium shape are represented by the following six deformed bosons which, together with b_c^\dagger , form a complete orthonormal basis,

$$\begin{aligned} b_u^\dagger &= (1 + R^2)^{-1/2} \left\{ -R s^\dagger + \frac{1}{\sqrt{2}} \left[p_{\rho,0}^\dagger - \frac{1}{\sqrt{2}} (p_{\lambda,1}^\dagger - p_{\lambda,-1}^\dagger) \right] \right\}, \\ b_v^\dagger &= \frac{1}{\sqrt{2}} \left[p_{\rho,0}^\dagger + \frac{1}{\sqrt{2}} (p_{\lambda,1}^\dagger - p_{\lambda,-1}^\dagger) \right], \\ b_w^\dagger &= \frac{1}{\sqrt{2}} \left[-\frac{1}{\sqrt{2}} (p_{\rho,1}^\dagger - p_{\rho,-1}^\dagger) + p_{\lambda,0}^\dagger \right], \\ b_x^\dagger &= \frac{1}{\sqrt{2}} (p_{\rho,1}^\dagger + p_{\rho,-1}^\dagger), \\ b_y^\dagger &= \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} (p_{\rho,1}^\dagger - p_{\rho,-1}^\dagger) + p_{\lambda,0}^\dagger \right], \\ b_z^\dagger &= \frac{1}{\sqrt{2}} (p_{\lambda,1}^\dagger + p_{\lambda,-1}^\dagger). \end{aligned} \quad (12)$$

The above boson operators represent excitations of the condensate which involve three vibrational intrinsic modes and three rotational Goldstone modes. The vibrational modes are a symmetric radial mode (b_u^\dagger), an antisymmetric radial mode (b_v^\dagger) and an angular mode (b_w^\dagger). The boson operators b_i^\dagger ($i = x, y, z$) are associated with rotational modes. They are obtained by applying the three components of the angular momentum operator, \hat{L}_i ($i = x, y, z$), on the condensate (8) with the shape parameters of eq. (10). The physical interpretation of these excitation modes are discussed in more detail in the following sections.

The vibrational and rotational excitations of a S_3 -invariant $U(7)$ mass operator can be studied by decomposing it into an intrinsic (vibrational) and a collective (rotational) part,

$$\hat{M}_{U(7)}^2 = \hat{M}_{\text{int}}^2 + \hat{M}_{\text{coll}}^2. \quad (13)$$

This resolution is obtained [14] by requiring that the intrinsic part annihilates the ground state condensate and has the same shape of the potential function as the original mass operator. The collective part of the mass operator has by construction a completely flat potential function. We now discuss each of these parts separately.

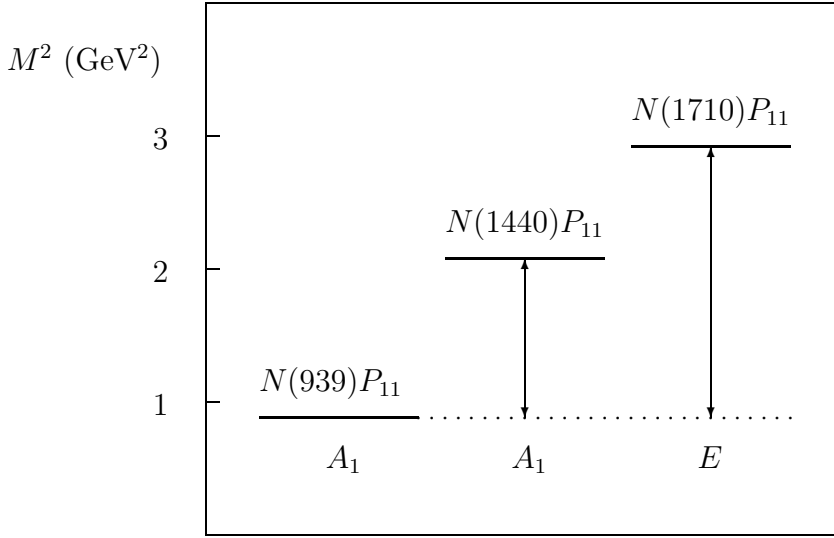


Figure 1: Schematic representation of the vibrational spectrum of nucleon resonances. The resonances are labeled by the usual spectroscopic notation [16] and their vibrational permutation symmetry.

VIBRATIONS

The intrinsic part of a S_3 -invariant $U(7)$ mass operator that annihilates the condensate with $r_\rho = r_\lambda = \bar{r} \neq 0$ and $\theta = \bar{\theta} = \pi/2$, consists of two terms only,

$$\begin{aligned} \hat{M}_{\text{int}}^2 = & \xi_1 \left(R^2 s^\dagger s^\dagger - p_\rho^\dagger \cdot p_\rho^\dagger - p_\lambda^\dagger \cdot p_\lambda^\dagger \right) \left(R^2 ss - \tilde{p}_\rho \cdot \tilde{p}_\rho - \tilde{p}_\lambda \cdot \tilde{p}_\lambda \right) \\ & + \xi_2 \left[\left(p_\rho^\dagger \cdot p_\rho^\dagger - p_\lambda^\dagger \cdot p_\lambda^\dagger \right) \left(\tilde{p}_\rho \cdot \tilde{p}_\rho - \tilde{p}_\lambda \cdot \tilde{p}_\lambda \right) + 4 \left(p_\rho^\dagger \cdot p_\lambda^\dagger \right) \left(\tilde{p}_\lambda \cdot \tilde{p}_\rho \right) \right], \quad (14) \end{aligned}$$

with $R^2 = 2\bar{r}^2$. The vibrational excitations can be obtained from a normal mode analysis of the intrinsic part of the mass operator. This is done [14] by rewriting the mass operator in terms of the deformed bosons of eqs. (11,12), replacing the condensate bosons, b_c and b_c^\dagger , by their classical mean field value, \sqrt{N} , and keeping terms of leading order in N . In the large N limit the normal modes are diagonal in the deformed boson basis and the mass operator reduces to a harmonic oscillator form,

$$\frac{1}{N} \hat{M}_{\text{int}}^2 = \lambda_1 b_u^\dagger b_u + \lambda_2 \left(b_v^\dagger b_v + b_w^\dagger b_w \right) + \mathcal{O}(1/\sqrt{N}), \quad (15)$$

with eigenvalues

$$\begin{aligned} \lambda_1 &= 4\xi_1 R^2, \\ \lambda_2 &= 4\xi_2 R^2 / (1 + R^2). \end{aligned} \quad (16)$$

Eq. (15) identifies the deformed bosons that correspond to the three fundamental vibrations: a symmetric stretching (u), an antisymmetric stretching (v) and a bending vibration (w). The first two are radial excitations, whereas the third is an angular mode which corresponds to oscillations in the angle θ between the two Jacobi coordinates. The angular mode is degenerate with the antisymmetric radial mode (with eigenvalue λ_2). This is in agreement with the point group classification of the fundamental vibrations for a symmetric X_3 configuration [15]. This shows that \hat{M}_{int}^2 describes the vibrational excitations of an oblate symmetric top. The rotational bosons, b_i^\dagger ($i = x, y, z$), are

missing from eq. (15) and correspond to massless Goldstone bosons associated with the rotation symmetry which is spontaneously broken in the condensate.

The intrinsic part of the mass operator describes the vibrational excitations of baryon resonances. In the large N limit the vibrational spectrum is harmonic,

$$M_{\text{vib}}^2 = N [\lambda_1 n_u + \lambda_2 (n_v + n_w)] . \quad (17)$$

In the nucleon sector, the vibrationless ground state ($n_u = n_v = n_w = 0$) is identified with the nucleon $N(939)$. The symmetric stretching vibration with frequency, λ_1 , has the same symmetry properties as the ground state. It is therefore possible to associate the vibration ($n_u = 1, n_v + n_w = 0$) with the Roper resonance $N(1440)$. The $N(1710)$ resonance could be a candidate for the other (two-dimensional) vibrational mode with ($n_u = 0, n_v + n_w = 1$). Similar considerations apply to the delta sector.

A great advantage of this analysis is that we have established a relation between the parameters of otherwise abstract ‘algebraic’ interactions, and measured experimental quantities, in this case the mass of specific resonances. The vibrational contribution to the masses, eq. (17), is obtained in the large N approximation which is expected to be valid in view of confinement. The extracted value of the parameters, ξ_1 and ξ_2 , can be used as a good starting value in a more detailed fitting procedure in an exact numerical calculation. In figure 1 we show a schematic representation of the vibrational spectrum of nonstrange baryon resonances.

ROTATIONS

On top of each vibrational excitation there is a whole series of rotational states. In a geometric description the rotational excitations are labeled by the angular momentum, L , its projection on the threefold symmetry axis, $K = 0, 1, \dots$, parity, and the transformation character under the permutation group. For a given value of K , the states have angular momentum $L = K, K + 1, \dots$, and parity $\pi = (-)^K$. For rotational states built on vibrations of type A_1 (symmetric under S_3) each L state is single for $K = 0$ and twofold degenerate for $K \neq 0$. For rotational states built on vibrations of type E (mixed S_3 symmetry) each L state is twofold degenerate for $K = 0$ and fourfold degenerate for $K \neq 0$. The transformation property of the states under the permutation group is found by multiplying the symmetry character of the vibrational and the rotational wave functions. The results are shown schematically in figure 2.

The collective part of the full mass operator contains the contribution of rotations to the mass. By construction \hat{M}_{coll}^2 consists of interaction terms which do not affect the shape of the potential function. Apart from the one-body term, \hat{n} , whose contribution to the mass is negligible for large N , the collective part of the mass operator can be expressed in terms of the Casimir operators of the following group chain,

$$U(7) \supset SO(7) \supset SO(6) \supset SO(3) \otimes SO(2) . \quad (18)$$

The bosonic Casimir invariants of $U(7)$ involve only the total number operator, $\hat{N} = \hat{n}_s + \hat{n}$, which is a conserved quantity. The only terms in the collective part of the mass operator that contribute to the excitation spectrum are therefore,

$$\hat{M}_{\text{coll}}^2 = \kappa_1 \hat{C}_{SO(7)} + \kappa_2 \hat{C}_{SO(6)} + \kappa_3 \hat{C}_{SO(3)} + \kappa_4 [\hat{C}_{SO(2)}]^2 . \quad (19)$$

Explicit expressions for the Casimir operators (\hat{C}_G) in terms of the generators of the groups (G) are shown in table 2.

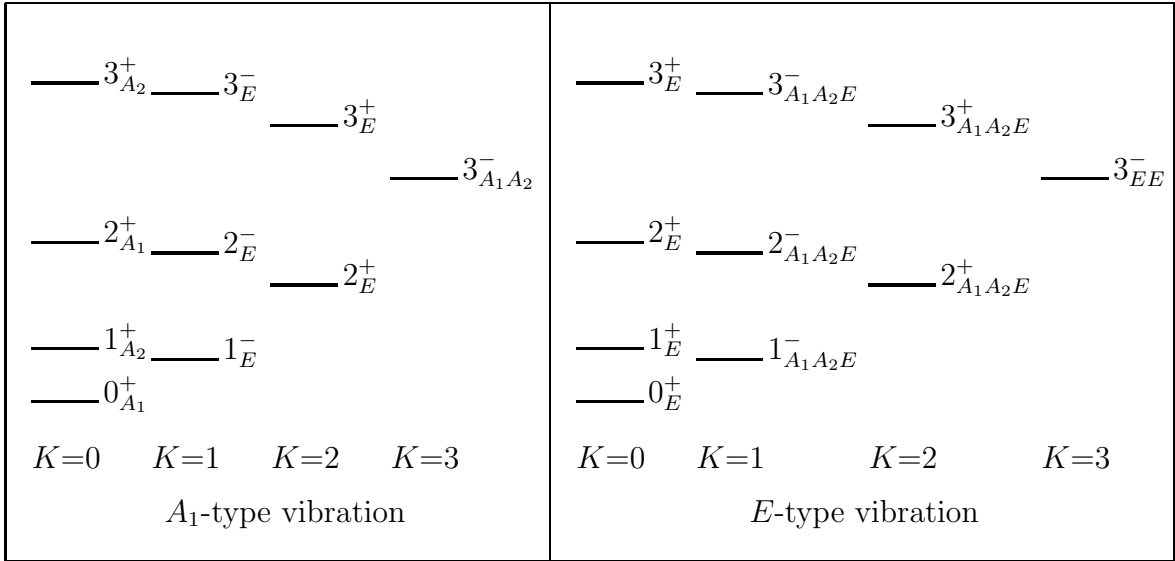


Figure 2: Schematic representation of the rotational structure built on vibrations of type A_1 (lefthand side) and vibrations of type E (righthand side). The levels are labeled by K , L_t^π , where t denotes the overall (vibrational plus rotational) permutation symmetry. Each E state is doubly degenerate.

The last two terms in the collective part of the mass operator (19) commute with any S_3 -invariant mass operator and thus correspond to *exact* symmetries. Their eigenvalues are similar in form to those of a symmetric top, $\kappa_3 L(L+1) + \kappa_4 K_y^2$. For the lowest eigenstates of each L the value of K_y coincides with that of the projection of the angular momentum on the threefold symmetry axis, denoted previously by K . Since L and K_y are always good quantum numbers, it is straightforward to include in the collective part of the mass operator higher orders of the corresponding Casimir operators. Generalizing the mass operator to contain arbitrary functions, $f(\hat{L} \cdot \hat{L})$ and $g(\hat{K}_y^2)$, the expression for the resulting rotational spectrum becomes $f[L(L+1)] + g[K_y^2]$.

In general the first two terms in eq. (19) do not correspond to exact symmetries of a S_3 -invariant mass operator and do not commute with the intrinsic part of the mass operator, eq. (14). Therefore, in addition to shifting and splitting the bands generated by \hat{M}_{int}^2 , they can also mix them. Their effect on the spectrum can be studied numerically. However, just as was done for the intrinsic part of the mass operator, we can gain physical insight of the different rotational terms in the collective part of the mass operator, by studying the large N limit,

$$\begin{aligned}
\frac{1}{N} \hat{M}_{\text{coll}}^2 = & -\eta_1 (b_u^\dagger - b_u)^2 - \eta_2 [(b_v^\dagger - b_v)^2 + (b_w^\dagger - b_w)^2] \\
& + \eta_3 [(b_x^\dagger + b_x)^2 + (b_z^\dagger + b_z)^2] - \eta_4 (b_y^\dagger - b_y)^2 + \mathcal{O}(1/\sqrt{N}) , \quad (20)
\end{aligned}$$

with

$$\begin{aligned}
\eta_1 &= \kappa_1 , \\
\eta_2 &= (1 + R^2)^{-1} [\kappa_1 + \kappa_2 R^2] , \\
\eta_3 &= (1 + R^2)^{-1} [\kappa_1 + (\kappa_2 + \frac{1}{2}\kappa_3) R^2] , \\
\eta_4 &= (1 + R^2)^{-1} [\kappa_1 + (\kappa_2 + \kappa_3 + \kappa_4) R^2] . \quad (21)
\end{aligned}$$

The b_x , b_y and b_z bosons are the Goldstone bosons connected with rotations in configuration space, whereas the b_u , b_v and b_w bosons correspond to generalized rotations in

Group G	Generators	Casimir operator \hat{C}_G
$SO(7)$	$G_S^{(1)}, G_\rho^{(1)}, G_\lambda^{(1)}, G_A^{(0)}, G_A^{(2)}, A_\rho, A_\lambda$	$\hat{C}_{SO(6)} + A_\rho \cdot A_\rho + A_\lambda \cdot A_\lambda$
$SO(6)$	$G_S^{(1)}, G_\rho^{(1)}, G_\lambda^{(1)}, G_A^{(0)}, G_A^{(2)}$	$G_S^{(1)} \cdot G_S^{(1)} + G_\rho^{(1)} \cdot G_\rho^{(1)} + G_\lambda^{(1)} \cdot G_\lambda^{(1)} + G_A^{(0)} \cdot G_A^{(0)} + G_A^{(2)} \cdot G_A^{(2)}$
$SO(3)$	$\hat{L} = \sqrt{2} G_S^{(1)}$	$\hat{L} \cdot \hat{L}$
$SO(2)$	$\hat{K}_y = -\sqrt{3} G_A^{(0)}$	\hat{K}_y

Table 2: Generators and Casimir operators of the groups relevant for the collective part of the mass operator.

a higher dimensional space. It is seen from eqs. (20,21) that the κ_1 -term of eq. (19) is associated with rotations in a six-dimensional space (u, v, w, x, y, z) space and the κ_2 -term with rotations in a five-dimensional subspace (v, w, x, y, z) . The κ_3 -term, $\hat{L} \cdot \hat{L}$, corresponds to ordinary three-dimensional (x, y, z) rotations and the κ_4 -term, \hat{K}_y^2 , is associated with a rotation about the threefold symmetry y -axis. The equality of the coefficients of the (v, w) and the (x, z) terms in eq. (20) is a reflection of the oblate-top nature of the S_3 -invariant mass operator.

Summarizing, we have shown that a S_3 -invariant $U(7)$ mass operator corresponds geometrically to rotations and vibrations of an equilateral equilibrium configuration of three quarks in a baryon. There are only six independent terms in the mass operator that determine the excitation spectrum: two for the vibrational excitations and four for the rotational excitations. The two vibrational terms cluster the states into bands. Two of the rotational terms correspond to exact symmetries of the mass operator and their eigenvalues are obtained in closed form. Whereas these terms are diagonal and hence only cause band splitting, the remaining two rotational terms may in addition cause band mixing.

BARYON RESONANCES

In the previous sections we introduced a $U(7)$ spectrum generating algebra for the spatial part of the baryon wave function. The total wave function for baryon resonances consists of a spatial, a spin-flavor and a color part,

$$\psi = \psi_L \phi_{sf} \psi_c. \quad (22)$$

To satisfy the Pauli principle for a system of identical quarks these parts have to be combined so that the total wave function is antisymmetric.

Both the spatial and the spin-flavor degrees of freedom contribute to the mass operator. In principle there could be also spin-flavor dependence in the coefficients of the interaction terms in \hat{M}_{space}^2 . Although these effects can be included without difficulty in an algebraic approach, in the present study we do not take them into

account and write

$$\hat{M}^2 = \hat{M}_{\text{space}}^2 + \hat{M}_{\text{spin-flavor}}^2 . \quad (23)$$

For the spin-flavor part we consider the $SU(6) \supset SU(3) \otimes SU(2)$ dynamic symmetry of Gürsey and Radicati [17]. In general, this symmetry may be broken (as is the case with the hyperfine interaction in the quark potential model), which would lead to coupling terms in eq. (23). Here, for simplicity, we limit ourselves to a diagonal breaking for which the eigenvalues are given in closed form,

$$M_{\text{spin-flavor}}^2 = a [\langle \hat{C}_{SU(6)} \rangle - 45] + b [\langle \hat{C}_{SU(3)} \rangle - 9] + c S(S+1) . \quad (24)$$

The first term involves the Casimir operator of the $SU(6)$ spin-flavor group with eigenvalues 45, 33 and 21 for the representations $56 \leftrightarrow A_1$, $70 \leftrightarrow E$ and $20 \leftrightarrow A_2$, respectively. The second term involves the Casimir invariant of the $SU(3)$ flavor group with eigenvalues 9 and 18 for the octet and decuplet, respectively. The last term contains the eigenvalues $S(S+1)$ of the total spin operator.

For the spatial part of the mass operator we take the S_3 invariant $U(7)$ mass operator discussed in the previous section which is decomposed into an intrinsic and a collective part

$$\hat{M}_{\text{space}}^2 = \hat{M}_{U(7)}^2 = \hat{M}_{\text{int}}^2 + \hat{M}_{\text{coll}}^2 . \quad (25)$$

Here, for simplicity, we take a simplified form for the collective part of the mass operator, containing only spatial rotations

$$\hat{M}_{\text{coll}}^2 \rightarrow \hat{M}_{\text{rot}}^2 = \alpha \sqrt{\hat{L} \cdot \hat{L} + \frac{1}{4}} , \quad (26)$$

with eigenvalues

$$M_{\text{rot}}^2 = \alpha (L + 1/2) . \quad (27)$$

The total mass operator can now be diagonalized numerically to get a fit for the masses of nonstrange baryon resonances. Instead, for the vibrational part we use the large N expression in eq. (17) with $R^2 = 1$, and obtain the following analytic expression for the masses,

$$M^2 = M_0^2 + M_{\text{vib}}^2 + M_{\text{rot}}^2 + M_{\text{spin-flavor}}^2 . \quad (28)$$

Here M_0^2 is a constant and the other contributions are given by eqs. (17,24,27).

For the nucleon which is a member of the flavor octet the relevant mass formulas read

$$\begin{aligned} M_N^2(A_1, L, S = 1/2) - M_{N(939)}^2 &= M_{\text{vib}}^2 + \alpha L , \\ M_N^2(E, L, S = 1/2) - M_{N(939)}^2 &= M_{\text{vib}}^2 + \alpha L - 12a , \\ M_N^2(E, L, S = 3/2) - M_{N(939)}^2 &= M_{\text{vib}}^2 + \alpha L - 12a + 3c , \\ M_N^2(A_2, L, S = 1/2) - M_{N(939)}^2 &= M_{\text{vib}}^2 + \alpha L - 24a . \end{aligned} \quad (29)$$

Here $M_{N(939)}^2 = 0.882 \text{ GeV}^2 = M_0^2 + \alpha/2 + 3c/4$. For the delta which is a member of the flavor decuplet the relevant mass formulas read

$$\begin{aligned} M_\Delta^2(A_1, L, S = 3/2) - M_{\Delta(1232)}^2 &= M_{\text{vib}}^2 + \alpha L , \\ M_\Delta^2(E, L, S = 1/2) - M_{\Delta(1232)}^2 &= M_{\text{vib}}^2 + \alpha L - 12a - 3c , \end{aligned} \quad (30)$$

Mass	Status	M_{exp}^2	J^π	L^π, K	S	t	M_{calc}^2	% Error	$(D, L_n^\pi)S$
$N(939)P_{11}$	****	0.882	$\frac{1}{2}^+$	$0^+, 0$	$\frac{1}{2}$	A_1	0.882	0	$(56, 0_0^+) \frac{1}{2}$
$N(1440)P_{11}$	****	2.074	$\frac{1}{2}^+$	$0^+, 0$	$\frac{1}{2}$	A_1	2.074	0	$(56, 0_2^+) \frac{1}{2}$
$N(1520)D_{13}$	****	2.310	$\frac{3}{2}^-$	$1^-, 1$	$\frac{1}{2}$	E	2.442	-5.7	$(70, 1_1^-) \frac{1}{2}$
$N(1535)S_{11}$	****	2.356	$\frac{1}{2}^-$	$1^-, 1$	$\frac{1}{2}$	E	2.442	-3.6	$(70, 1_1^-) \frac{1}{2}$
$N(1650)S_{11}$	****	2.722	$\frac{1}{2}^-$	$1^-, 1$	$\frac{3}{2}$	E	2.817	-3.5	$(70, 1_1^-) \frac{3}{2}$
$N(1675)D_{15}$	****	2.806	$\frac{5}{2}^-$	$1^-, 1$	$\frac{3}{2}$	E	2.817	-0.4	$(70, 1_1^-) \frac{3}{2}$
$N(1680)F_{15}$	****	2.822	$\frac{5}{2}^+$	$2^+, 0$	$\frac{1}{2}$	A_1	2.994	-6.0	$(56, 2_2^+) \frac{1}{2}$
$N(1700)D_{13}$	***	2.890	$\frac{3}{2}^-$	$1^-, 1$	$\frac{3}{2}$	E	2.817	2.5	$(70, 1_1^-) \frac{3}{2}$
$N(1710)P_{11}$	***	2.924	$\frac{1}{2}^+$	$0^+, 0$	$\frac{1}{2}$	E	2.924	0	$(70, 0_2^+) \frac{1}{2}$
$N(1720)P_{13}$	****	2.958	$\frac{3}{2}^+$	$2^+, 0$	$\frac{1}{2}$	A_1	2.994	-1.2	$(56, 2_2^+) \frac{1}{2}$
$N(2190)G_{17}$	****	4.796	$\frac{7}{2}^-$	$3^-, 1$	$\frac{1}{2}$	E	4.554	5.0	$(70, 3_3^-) \frac{1}{2}$
					$\frac{3}{2}$	E	4.929	-2.8	
$N(2220)H_{19}$	****	4.928	$\frac{9}{2}^+$	$4^+, 0$	$\frac{1}{2}$	A_1	5.106	-3.6	$(56, 4_4^+) \frac{1}{2}$
$N(2250)G_{19}$	****	5.063	$\frac{9}{2}^-$	$3^-, 1$	$\frac{3}{2}$	E	4.929	2.6	$(70, 3_3^-) \frac{3}{2}$
$N(2600)I_{1,11}$	***	6.760	$\frac{11}{2}^-$	$5^-, 1/5$	$\frac{1}{2}$	E	6.666	1.4	

Table 3: Oblate top classification of nonstrange baryons of the N family with $I = 1/2$. Here t denotes the overall permutation symmetry. The last column lists the dominant representation of the quark model assignment in a $SU_{sf}(6) \otimes O(3)$ basis [16]. M^2 is given in GeV^2 . The experimental values are taken from [16].

and $M_{\Delta(1232)}^2 = 1.518 \text{ GeV}^2 = M_{N(939)}^2 + 9b + 3c$. The parameters M_0^2 , b , $\xi_1 N$ and $\xi_2 N$ are determined from the mass squared of $N(939)$, $\Delta(1232)$, $N(1440)$ and $N(1710)$ respectively. The remaining parameters a , c , and α are determined by fitting the mass of the baryon resonances with *** or **** status. The values of the parameters (in GeV^2) extracted in a least square fit are

$$\begin{aligned}
M_0^2 &= 0.260, \quad \xi_1 N = 0.298, \quad \xi_2 N = 0.769, \quad \alpha = 1.056, \\
a &= -0.042, \quad b = 0.029, \quad c = 0.125.
\end{aligned}
\tag{31}$$

Mass	Status	M_{exp}^2	J^π	L^π, K	S	t	M_{calc}^2	% Error	$(D, L_n^\pi)S$
$\Delta(1232)P_{33}$	****	1.518	$\frac{3}{2}^+$	$0^+, 0$	$\frac{3}{2}$	A_1	1.518	0	$(56, 0_0^+) \frac{3}{2}$
$\Delta(1620)S_{31}$	****	2.624	$\frac{1}{2}^-$	$1^-, 1$	$\frac{1}{2}$	E	2.703	-3.0	$(70, 1_1^-) \frac{1}{2}$
$\Delta(1700)D_{33}$	****	2.890	$\frac{3}{2}^-$	$1^-, 1$	$\frac{1}{2}$	E	2.703	6.5	$(70, 1_1^-) \frac{1}{2}$
$\Delta(1900)S_{31}$	***	3.610	$\frac{1}{2}^-$	$1^-, 1$	$\frac{1}{2}$	E	3.895	-7.9	
$\Delta(1905)F_{35}$	****	3.629	$\frac{5}{2}^+$	$2^+, 0$	$\frac{3}{2}$	A_1	3.630	-0.03	$(56, 2_2^+) \frac{3}{2}$
$\Delta(1910)P_{31}$	****	3.648	$\frac{1}{2}^+$	$2^+, 0$	$\frac{3}{2}$	A_1	3.630	0.5	$(56, 2_2^+) \frac{3}{2}$
$\Delta(1920)P_{33}$	***	3.686	$\frac{3}{2}^+$	$2^+, 0$	$\frac{3}{2}$	A_1	3.630	1.5	$(56, 2_2^+) \frac{3}{2}$
$\Delta(1930)D_{35}$	***	3.725	$\frac{5}{2}^-$	$2^-, 1$	$\frac{1}{2}$	E	3.759	-0.9	
$\Delta(1950)F_{37}$	****	3.803	$\frac{7}{2}^+$	$2^+, 0$	$\frac{3}{2}$	A_1	3.630	4.5	$(56, 2_2^+) \frac{3}{2}$
$\Delta(2420)H_{3,11}$	****	5.856	$\frac{11}{2}^+$	$4^+, 0$	$\frac{3}{2}$	A_1	5.742	1.9	$(56, 4_4^+) \frac{3}{2}$

Table 4: Oblate top classification of nonstrange baryons of the Δ family with $I = 3/2$. For further information see table 3.

In table 3 and 4 we show the fit to the nonstrange baryon masses of the nucleon and the delta family, respectively. These results obtained in the large N limit are substantiated by exact numerical calculations. We find a reasonable overall fit for the *** and **** resonances with an r.m.s. deviation of $\delta_{\text{rms}} = 0.14 \text{ GeV}^2$. Some characteristic features of the fit are:

(i) All experimentally well established resonances are reproduced by the calculations. Experimentally known but uncertain resonances with ** status can be accommodated in the fit as well. For example, for the $\Delta(1600)P_{33}$ resonance with $M_{\text{exp}}^2 = 2.56 \text{ GeV}^2$ we find $M_{\text{calc}}^2 = 2.71 \text{ GeV}^2$.

(ii) In the present calculation most states listed in the tables are rotational members of the ground band ($n_u = 0, n_v + n_w = 0$). We have associated $N(1440)$ and $\Delta(1900)$ with A_1 vibrational bands ($n_u = 1, n_v + n_w = 0$), and $N(1710)$ with the E vibrational band ($n_u = 0, n_v + n_w = 1$).

(iii) The oblate top assignments of orbital angular momentum, spin and permutation symmetry are similar to the quark model assignment.

(iv) The low-lying nucleon resonances, P_{11} , D_{13} , S_{11} , the cluster of $I = 1/2$ resonances in the mass range 1.6-1.7 GeV and the cluster of $I = 3/2$ resonances near 1.9 GeV are well reproduced. Above this mass range there are many more resonances predicted than observed experimentally. This is due to the fact that we have associated the low-lying $N(1440)$ and $N(1710)$ resonances with vibrational bandheads. Consequently the

rotational states built on top of them occur low in the mass spectrum. This problem of missing resonances is known to exist in quark potential models as well.

One possible explanation is that the ‘missing’ resonances indeed do exist, but that they cannot be resolved individually since there are many overlapping resonances in that mass region. Another explanation could be that these resonances are decoupled from the πN channel [18]. Since at present most experimental information is from pion scattering, they could simply not be excited in this type of experiments. Future experiments with electromagnetic probes may shed more light on this question. It has also been suggested that the Roper and the $N(1710)$ are hybrid states [19]. If that were the case these resonances are outside the $U(7)$ model space and therefore the coefficients, ξ_1 and ξ_2 , in the intrinsic part of the mass operator cannot be determined from the present data. A large value of ξ_1 , ξ_2 would shift the vibrational bands up in mass, without affecting the rotational members of the ground band.

Finally, we note that in the $U(7)$ model we have obtained a fit to the nonstrange baryon masses which is comparable to that in quark potential models [6, 7], although the underlying quark dynamics is quite different. This shows that the masses alone are not sufficient to distinguish between different forms of quark dynamics, *e.g.* single-particle *vs.* collective motion.

SUMMARY

In this contribution we have proposed to use $U(7)$ as a spectrum generating algebra for a geometry-oriented description of baryons. It is combined with the spin-flavor and color parts into a $U(7) \otimes SU_{sf}(6) \otimes SU_c(3)$ SGA for baryon spectroscopy. Although we have limited the discussion to nonstrange baryons, the $U(7)$ model can accommodate strange baryons as well. The present model allows one to study both collective-like and single-particle-like motion in a single algebraic framework. Collectivity corresponds to coherent and strongly correlated motion of quarks which requires a strong coupling of harmonic oscillator shells. We have shown explicitly that there exists $U(7)$ mass operators that strongly mix states with different oscillator quanta, but still preserve the permutation symmetry.

We have applied the model to the family of nucleon and delta resonances and found good overall agreement with the observed masses. The fit is of comparable quality to that obtained in quark potential models. In addition to mass spectra, the model provides wave functions which can be used to calculate other observables such as helicity amplitudes and (transition) form factors. These quantities provide a far more sensitive test to details in the wave functions than the mass spectrum. We are in the process of examining a variety of such observables in the $U(7)$ model. Our goal is to identify signatures which may distinguish single-particle from collective aspects of quark dynamics in baryons, as well as to provide guidance to future experiments.

ACKNOWLEDGEMENTS

On the occasion of his 50th birthday, it is gratifying and most proper to dedicate this contribution to F. Iachello who is actively involved in advancing the methods and ideas reported in this work.

REFERENCES

- [1] See *e.g.* M. Gell-Mann and Y. Ne'eman, 'The eightfold way', W.A. Benjamin (1964).
- [2] S. Okubo, Progr. Theor. Phys. **27**, 949 (1962).
- [3] F. Iachello, Nucl. Phys. **A497**, 23c (1989); *ibid.* **A518**, 173 (1990); *ibid.* Proc. of the '4th workshop on perspectives in nuclear physics at intermediate energies', S. Boffi, C. Ciofi degli Atti and M. Giannini, eds., World Scientific, Singapore (1989), page 17.
- [4] F. Iachello and D. Kusnezov, Phys. Lett. **255B**, 493 (1991); *ibid.* Phys. Rev. **D45**, 4156 (1992).
- [5] F. Iachello, N.C. Mukhopadhyay and L. Zhang, Phys. Lett. **256B**, 295 (1991); *ibid.* Phys. Rev. **D44**, 898 (1991).
- [6] N. Isgur and G. Karl, Phys. Rev. **18**, 4187 (1978); *ibid.* **D19**, 2653 (1979); *ibid.* **D20**, 1191 (1979); K.-T. Chao, N. Isgur and G. Karl, Phys. Rev. **D23**, 155 (1981).
- [7] S. Capstick and N. Isgur, Phys. Rev. **D34**, 2809 (1986).
- [8] R. Bijker, A.E.L. Dieperink and A. Leviatan, these proceedings.
- [9] M. Moshinsky, in 'The Harmonic Oscillator in Modern Physics: From Atoms to Quarks', Gordon and Breach, 1969 and references therein.
- [10] A. Arima and F. Iachello, Phys. Rev. Lett. **35**, 1069 (1975);
F. Iachello and A. Arima, 'The Interacting Boson Model' (Cambridge Univ. Press, Cambridge, 1987).
- [11] F. Iachello, Chem. Phys. Lett. **78**, 581 (1981).
- [12] K.C. Bowler, P.J. Corvi, A.J.G. Hey, P.D. Jarvis and R.C. King, Phys. Rev. **D24**, 197 (1981); A.J.G. Hey and R.L. Kelly, Phys. Rep. **96**, 71 (1983).
- [13] R. Gilmore and D.H. Feng, Nucl. Phys. **A301**, 189 (1978).
- [14] A. Leviatan and M.W. Kirson, Ann. Phys. **188**, 142 (1988);
A. Leviatan, J. Chem. Phys. **91**, 1706 (1989).
- [15] G. Herzberg, 'Molecular Spectra and Molecular Structure II: Infrared and Raman Spectra of Polyatomic Molecules' (Van Nostrand Reinhold, Princeton, NJ 1945).
- [16] Particle Data Group, Phys. Lett. **B239**, 1 (1990).
- [17] F. Gürsey and L.A. Radicati, Phys. Rev. Lett. **13**, 173 (1964).
- [18] R. Koniuk and N. Isgur, Phys. Rev. **D21**, 1868 (1980).
- [19] Z. Li, Phys. Rev. **D44**, 2841 (1991);
Z. Li, V. Burkert and Z. Li, Phys. Rev. **D46**, 70 (1992).